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M.Sc. (Sem. - III) Examination October - 2025

Mathematics (Functional Analysis - I)

Time: 3 Hours] **[Total Marks: 70**

સૂચના : / Instructions	Seat No.:
(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book (2) Answer all questions (3) Figures to the right indicate marks (4) Follow usual notations and conventions	Student's Signature

Q. 1 Attempt Any Two: **14**

(a) Prove that dual space of l^p is l^q ; where $1 < p < \infty$ and q is the conjugate of p. i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

(b) Let $\{x_{1,} x_2, ..., x_n\}$ be the linearly independent set of vectors in a normed space X (of any dimension). Then prove that there exists $C > 0$ such that for every choice of scalars $\propto_{1,} \propto_{2, ...} \propto_n$; $|| \sum_{i=1}^{n} \alpha_i x_i || \geq C \sum_{i=1}^{n} |\alpha_i|$

(c) State and Prove Holder's inequality.

Q. 2 Attempt Any Two: **14**

(a) A dot product defines a functional $f: R^3 \to$ R by $f(x) = x.\ a = x_1 \propto_1 + x_2 \propto_2 + x_3 \propto_3$; where $x \in R^3$, $a = (\propto_j) \in R^3$ is fixed. Then show that f is linear, bounded with $\| f \| = \| a \|$.

(b) Prove that the vector space $B(X, Y)$ of all bounded linear operator from a normed space X into a normed space Y itself is a normed space with norm

defined by $\| T \| = \underset{\substack{x \in X \\ x \neq 0}}{Sup} \frac{|| Tx ||}{|| x ||} = \underset{\substack{x \in X \\ || x || = 1}}{Sup} || Tx ||$

(c) If Y is a Banach space then prove that B (X, Y) is a Banach space.

Q. 3 Attempt Any Two: **14**

(a) Let H be a Hilbert space and $Y \subset H$, then prove that the orthogonal complement $Y^{\perp}$ of Y is a closed linear subspace of H.

(b) Space L^2 $[a, b]$ with an inner product defined as $\langle x, y\rangle = \int_a^b x(t)\overline{y(t)}\,dt$ and norm on L^2 $[a, b]$ is defined as $\|x\| = \left(\int_a^b |x(t)|^2\, dt\right)^{1/2}$ .Prove that L^2 $[a, b]$ is a Hilbert space.

(c) Let X be an inner product space and let M be a complete convex subspace Y. Let $x \in X$ be fixed, then prove that is $z = x - y$ orthogonal to Y.

Q. 4 Attempt Any Two: **14**

(a) Prove that $C[a, b]$ is complete space.

(b) Define a metric on C^n and prove that it is a metric space.

(c) Define equivalent norms. Prove that on a finite dimensional vector space X, any norm $\|.\|$ is equivalent to any other norm $\|.\|_0$.

Q. 5 Attempt Any Two: **14**

(a) Let Y and Z be subspaces of a normed space X (of any dimension) and suppose Y is a closed and is a proper subset of Z, then prove that for every real, there exists $z \in Z$ such that $\|z\| = 1$ and $\|z - y\| \geq \theta$ for every $y \in Y$.

(b) Show that $\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$ and $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$, defined on a vector space of ordered n-tuples of numbers are equivalent.

(c) Let Y be any closed subspace of a Hilbert space H. Then prove that $H = Y \oplus Z$; where $Z = Y^\perp$.